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Moduli of Simple Holomorphic Pairs and Effective Divisors

By S. KOSAREW and P. LUPASCU

Abstract. In this note we identify two complex structures (one is given by algebraic geometry, the other by gauge theory) on the set of isomorphism classes of holomorphic bundles with section on a given compact complex manifold.

In the case of *line* bundles, these complex spaces are shown to be isomorphic to a space of effective divisors on the manifold.

Introduction

Let $(X, \mathcal{O}_X)$ be a compact complex analytic space. We denote by $\mathrm{Div}^+(X)$ the space of all Cartier divisors (including the empty one) on X. This set is a Zariski open subspace in the entire Douady space $\mathcal{D}(X)$, parameterizing *all* compact subspaces of X. If X is smooth (or more generally *locally factorial*), then $\mathrm{Div}^+(X)$ is a union of connected components of $\mathcal{D}(X)$ [9].

We consider pairs $(\mathcal{E}, \phi)$ consisting of an invertible sheaf $\mathcal{E}$ over X coupled with a holomorphic section ϕ in $\mathcal{E}$, which is locally a non zero divisor. Two such pairs $(\mathcal{E}_i, \phi_i)$ are called equivalent if there exists an isomorphism of sheaves $\Theta : \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ such that $\Theta \circ \phi_1 = \phi_2$. The set of equivalence classes is called the moduli space of simple holomorphic pairs of rank one on X. This moduli space can be given a structure of a complex analytic space using standard results of deformation theory (see [11]).

There exists a natural bijective map from the moduli space of simple holomorphic pairs of rank one into $\mathrm{Div}^+(X)$ sending the equivalence class of a pair $(\mathcal{E}, \phi)$ into the divisor, given by the vanishing locus of the section ϕ. In the first part of the paper, we prove that this one-to-one correspondence is in fact an analytic isomorphism with respect to these natural structures on the two spaces.

If X is a *smooth* compact complex manifold, there is a second possibility of defining an analytic structure on the moduli space of simple holomorphic pairs, namely by using gauge-theoretical methods (compare [13], [14]). In the last part

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we show that, in this case, these two structures are isomorphic. This is the "pair"-version of a previous result due to MIYAJIMA [12] in the case of moduli spaces of simple *bundles*.

Throughout this paper we adopt the following notations:

(*sets*)	:	the category of sets;
(*an*)	:	the category of (not necessarily reduced) complex spaces;
(*an*/R)	:	the category of relative complex spaces over a complex space R;
(*germs*)	:	the category of germs of complex spaces;
h_S	:	the canonical contravariant functor $h_S : \mathcal{C} \longrightarrow$ (*sets*), $h_S = \mathrm{Hom}(\,\cdot\,, S)$ associated to an object S belonging to a category $\mathcal{C}$.

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1 Main result

The purpose of this part is to prove the following

Theorem 1.1. *Let X be a compact complex space with $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$. Then there exists a natural complex analytic isomorphism between the moduli space of simple holomorphic pairs of rank one and the space* $\mathrm{Div}^+(X)$.

Remark. Let $(X, \mathcal{O}_X)$ be a compact complex space with $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$. Then:
i.) The Picard functor $\underline{\mathrm{Pic}}_X$ is representable (cf. [2], p.337).
ii.) There exists a Poincaré line bundle over $X \times \mathrm{Pic}(X)$.

The existence of a Poincaré bundle can be seen as follows (see [5], p.55 in the smooth case). The Leray spectral sequence for the projection morphism $\pi : X \times S \longrightarrow S$ leads to the exact sequence

$$0 \to \mathrm{Pic}(S) \to \mathrm{Pic}(X \times S) \to H^0(\mathcal{R}^1\pi_*\mathcal{O}^*_{X\times S}) \to H^2(\pi_*\mathcal{O}^*_{X\times S}) \to H^2(\mathcal{O}^*_{X\times S}) \,.$$

Since $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$, the last morphism is *injective*, hence one gets an exact sequence

$$0 \to \mathrm{Pic}(S) \to \mathrm{Pic}(X \times S) \to H^0(\mathcal{R}^1\pi_*\mathcal{O}^*_{X\times S}) \cong \mathrm{Hom}(S, \mathrm{Pic}(X)) \to 0 \,. \quad (1)$$

For $S = \mathrm{Pic}(X)$, this sequence leads to a line bundle $\mathcal{P}$ on $X \times \mathrm{Pic}(X)$ such that $\mathcal{P}|_{X\times\{[\xi]\}} \cong \xi$ for every $[\xi] \in \mathrm{Pic}(X)$. □

A complex structure on the space of simple holomorphic pairs is given as follows:

Let $\mathcal{P}$ be a Poincaré bundle on $X \times \mathrm{Pic}(X)$. By [8] there exists a linear fiber space H over $\mathrm{Pic}(X)$, which represents the functor $\mathcal{H} : (an/\mathrm{Pic}(X)) \longrightarrow$ (*sets*) given by

$$\Big(S \xrightarrow{f} \mathrm{Pic}(X)\Big) \longmapsto \mathrm{Hom}\Big(\mathcal{O}_{X\times S}, (\mathrm{id}_X \times f)^*\mathcal{P}\Big) \,.$$

(The action of $\mathcal{H}$ on morphisms is given by "pull-back".) In particular, for every complex space S over $\mathrm{Pic}(X)$, there is a bijection

$$\mathrm{Hom}\Big(\mathcal{O}_{X\times S}, (\mathrm{id}_X \times f)^*\mathcal{P}\Big) \cong \mathrm{Hom}_{\mathrm{Pic}(X)}(S, H)\,. \tag{2}$$

Let $\tilde{O} \subset H$ be the subset consisting of zero divisors. The multiplicative group $\mathbb{C}^*$ operates on $H' := H \setminus \tilde{O}$ such that the quotient $\mathrm{P}(H') := H'/\mathbb{C}^*$ becomes an open subset of a projective fiber space over $\mathrm{Pic}(X)$. The fiber over $[\xi] \in \mathrm{Pic}(X)$ can be identified with an open subset of $PH^0(X, \xi)$. Moreover, $\mathrm{P}(H')$ coincides set-theoretically with the moduli space of simple holomorphic pairs, defining a natural analytic structure on it.

In order to prove that $\mathrm{P}(H')$ and $\mathrm{Div}^+(X)$ are isomorphic as complex spaces, it suffices to prove that the associated functors $h_{\mathrm{P}(H')}$ and $h_{\mathrm{Div}^+(X)}$ are isomorphic. More precisely, we show that both are isomorphic to the contravariant functor

$$F : (an) \longrightarrow (sets),$$

defined by $S \longmapsto F(S)$, where $F(S)$ denotes the equivalence classes of pairs $(\mathcal{E}, \phi)$, where $\mathcal{E}$ is an invertible sheaf on $X \times S$, and ϕ is a holomorphic section in $\mathcal{E}$ whose restriction to each fiber $X \times \{s\}$ is locally a non zero divisor. (Two such tuples are called equivalent if there exists an isomorphism of sheaves $\Theta : \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ such that $\Theta \circ \phi_1 = \phi_2$.)

Remark. Note that if $(\mathcal{E}, \phi) \overset{\Theta}{\sim} (\mathcal{E}, \phi)$, then necessarily $\Theta = \mathrm{id}_{\mathcal{E}}$. Using this simple observation, it is easy to see that the functor F is of local nature (it is a sheaf), i.e. given any complex space S together with an open covering $S = \cup S_i$, the following sequence is exact:

$$F(S) \longrightarrow \prod F(S_i) \longrightarrow \prod F(S_i \cap S_j)\,.$$

Recall that a *relative Cartier divisor* in $X \times S$ over S is a Cartier divisor $Z \subset X \times S$ which is flat over S. We denote by $\mathrm{Div}^+_S(X)$ the set of all relative Cartier divisors (including the empty one) over a fixed S.

Lemma 1.2. *Let S be a fixed complex space. The map*

$$Z : F(S) \longrightarrow \mathrm{Div}^+_S(X)$$

sending the class of $(\mathcal{E}, \phi)$ into the Cartier divisor given by the vanishing locus of the section ϕ is well defined and bijective.

Proof. It is clear that the vanishing locus $Z := Z(\phi) \subset X \times S$ depends only on the isomorphism class of $(\mathcal{E}, \phi)$. We need to show that Z is flat over S. Take $(x, s) \in Z$, denote $A := \mathcal{O}_{S,s}$, $B := \mathcal{O}_{X\times S,(x,s)}$ and let $\mathfrak{m} \subset A$ be the maximal ideal. Since the section ϕ restricted to each fiber $X \times \{s\}$ is locally a non zero divisor, we have $\mathcal{O}_{Z,(x,s)} = B/uB$ for some nonzero divisor u. The flatness of the morphism $A \longrightarrow B/uB$ follows then from the general Bourbaki-Grothendieck criterion [7], p. 152 since

$$\mathrm{Tor}_1^A(A/\mathfrak{m}, B/uB) = \ker(B/\mathfrak{m}B \xrightarrow{\cdot u} B/\mathfrak{m}B) = \{0\}.$$

Take two simple pairs $(\mathcal{E}_i, \phi_i)$ over $X \times S$, defining the same divisor $Z := Z(\phi_1) = Z(\phi_2)$. The invertible sheaf $\mathcal{O}_{X\times S}(Z)$ admits a canonical section ϕ_{can}, and both pairs $(\mathcal{E}_i, \phi_i)$ are equivalent to the pair $(\mathcal{O}_{X\times S}(Z), \phi_{\text{can}})$. This proves the injectivity.

Given $Z \in \mathrm{Div}_S^+(X)$, the associated canonical section ϕ_{can} restricted to each fiber $X \times \{s\}$ is locally a non zero divisor since Z lies flat over S. This gives the surjectivity. □

The above lemma shows that there exists a natural isomorphism of functors between F and the functor $G : (an) \longrightarrow (sets)$,

$$G(S) := \mathrm{Div}_S^+(X) \subset \mathcal{D}(X \times S) .$$

It follows from the general result of Douady [6] that G is a representable functor and its representation space is exactly $\mathrm{Div}^+(X)$. In order to prove Theorem 1.1 it suffices to show the following

Theorem 1.3. *The functor F is representable by the space of simple holomorphic pairs* $\mathrm{P}(H')$.

Proof. Since F is a sheaf, it suffices to prove that F and $h_{\mathrm{P}(H')}$ are isomorphic as functors defined on the category of germs of analytic spaces. The following observation shows that $h_{\mathrm{P}(H')}$ is isomorphic to the sheafified functor associated to the quotient functor $h_{H'}/h_{\mathbb{C}^*}$:

Lemma 1.4. *Let M be a complex analytic space, and let G be a complex Lie group acting smoothly and freely on M. Suppose that the quotient M/G exists in the category of analytic spaces such that the canonical morphism $M \longrightarrow M/G$ is smooth. Then the canonical morphism of functors*

$$(h_M/h_G)^{\#} \longrightarrow h_{M/G} . \tag{3}$$

is an isomorphism (The superscript $^{\#}$ denotes here the associated sheafified functor.)

Proof. Since the projection $M \longrightarrow M/G$ is smooth, one has an epimorphism of sheaves $h_M \longrightarrow h_{M/G}$, hence the morphism (3) is also an epimorphism.

Furthermore, since G acts smoothly and freely on M, the morphism $G \times M \longrightarrow M \times_{M/G} M$, $(g, m) \longmapsto (m, gm)$ is an *isomorphism* by the relativ implicit function theorem. This shows that (3) is also a monomorphism, i.e. an isomorphism. □

Let $p : H' \longrightarrow \mathrm{Pic}(X)$ be the natural morphism and consider the corresponding tautological homomorphism

$$u : \mathcal{O}_{X\times H'} \longrightarrow (\mathrm{id}_X \times p)^*\mathcal{P}$$

given by the bijection (2).

The morphism of functors $h_{H'} \longrightarrow F$ is defined by sending $\phi : S \longrightarrow H'$ to the isomorphism class of the simple pair

$$\mathcal{O}_{X\times S} \xrightarrow{\phi^* u} (\mathrm{id}_X \times p \circ \phi)^*\mathcal{P} .$$

(Note that $\phi^* u|_{X\times\{s\}} = \phi(s) \in H'$ for every $s \in S$.)

<u>Injectivity:</u> Let ϕ_1, ϕ_2 be two morphisms from S to H' such that the associated simple pairs are isomorphic. It follows in particularly that the two sheaves $(\mathrm{id}_X \times p \circ \phi_i)^*\mathcal{P}$ are isomorphic via some Θ. The sequence (1) implies $p \circ \phi_1 = p \circ \phi_2$. The isomorphism Θ becomes an automorphism, and is given by multiplication with some element $a_\Theta \in H^0(X \times S, \mathcal{O}^*_{X \times S}) \cong H^0(S, \mathcal{O}^*_S)$ (since $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$). It follows that ϕ_1, ϕ_2 are conjugate under the action of $a_\Theta : S \longrightarrow \mathbb{C}^*$, i.e. the morphism

$$(h_{H'}/h_{\mathbb{C}^*})^\# \longrightarrow F$$

is injective.

<u>Surjectivity:</u> Consider a germ $(S, 0)$ of complex space and a simple pair $(\mathcal{O}_{X \times S} \xrightarrow{\alpha} \mathcal{E})$. The corresponding morphism $f \in \mathrm{Hom}_{\mathrm{Pic}(X)}(S, H')$ has the property

$$\mathcal{E} \cong (\mathrm{id}_X \times p \circ f)^*\mathcal{P} \otimes \mathcal{L}$$

for some $[\mathcal{L}] \in \mathrm{Pic}(S)$. However, since we are working on the germ S, we may assume that $\mathcal{L}$ is trivial. In this way, we obtain a simple pair

$$(\mathcal{O}_{X \times S} \xrightarrow{\alpha'} (\mathrm{id}_X \times p \circ f)^*\mathcal{P}),$$

which by (2) leads to a morphism from S to H'. This proves the surjectivity, and completes the proof of theorem 1.3. □

Corollary 1.5. *Let $(X, \mathcal{O}_X)$ be a compact, reduced, connected and locally irreducible analytic space, and let $m \in H^2(X, \mathbb{Z})$ be a fixed cohomology class. Consider the closed subspace of* $\mathrm{Div}^+(X)$ *given by*

$$\mathrm{Dou}(m) := \left\{ Z \in \mathrm{Div}^+(X) \;\middle|\; c_1(\mathcal{O}_X(Z)) = m \right\}.$$

Then there exists an isomorphism of complex spaces

$$\mathrm{Dou}(m) \cong \left\{ (\mathcal{E}, \phi) \;\middle|\; \begin{array}{l} \mathcal{E} \text{ invertible sheaf on } X, \\ c_1(\mathcal{E}) = m,\ 0 \neq \phi \in H^0(X, \mathcal{E}) \end{array} \right\} \Big/ \sim \ .$$

Proof. One has a commutative diagram

$$\begin{array}{ccc} \mathrm{P}(H') & \xrightarrow{\cong} & \mathrm{Div}^+(X) \\ & \searrow \quad \swarrow & \\ & \mathrm{Pic}(X) & \end{array}$$

where the vertical arrows are the natural projective morphisms. The assertion follows by taking the analytic pull-back of the component $\mathrm{Pic}^m(X) \subset \mathrm{Pic}(X)$ via these two maps. □

Remark. If X is a smooth manifold, then $\mathrm{Dou}(m)$ is a union of connected components of $\mathrm{Div}^+(X)$. Moreover, if X admits a Kähler metric, the spaces $\mathrm{Dou}(m)$ are always compact. This follows from Bishop's compactness theorem, since all divisors in $\mathrm{Dou}(m)$ have the same volume (with respect to any Kähler metric).

This property fails in the case of manifolds which do not allow Kähler metrics, since a non-Kählerian manifold may have (nonempty) effective divisors which are homologically trivial. (Take for instance a (elliptic) surface X with $H^2(X, \mathbb{Z}) = 0$.)

2 Gauge-theoretical point of view

When X is a smooth compact, connected complex manifold it is possible to construct a "gauge-theoretical" moduli space of simple holomorphic pairs (of any rank) on X (compare [13], [14]).

Let E be a fixed $\mathcal{C}^\infty$ complex vector bundle of rank r on X. We recall the following basic facts from complex differential geometry:

Definition 2.1. A *semiconnection* (of type (0,1)) in E is a differential operator $\bar{\delta}$: $A^0(E) \longrightarrow A^{0,1}(E)$ satisfying the Leibniz rule

$$\bar{\delta}(f \cdot s) = \bar{\partial}(f) \otimes s + f \cdot \bar{\delta}(s) \qquad \forall\, f \in \mathcal{C}^\infty(X, \mathbb{C}), s \in A^0(E)\ .$$

The space of all semiconnections in E will be denoted by $\bar{\mathcal{D}}(E)$; it is an affine space over $A^{0,1}(\operatorname{End} E)$. Every $\bar{\delta} \in \bar{\mathcal{D}}(E)$ admits a natural extension

$$\bar{\delta} : A^{p,q}(E) \longrightarrow A^{p,q+1}(E)$$

such that

$$\bar{\delta}(\alpha \otimes s) = \bar{\partial}(\alpha) \otimes s + (-1)^{p+q} \alpha \wedge \bar{\delta}(s) \qquad \forall\, \alpha \in A^{p,q}(X), s \in A^0(E)\ .$$

Moreover, $\bar{\delta}$ induces $\bar{D} : A^{p,q}(\operatorname{End} E) \longrightarrow A^{p,q+1}(\operatorname{End} E)$ by

$$\bar{D}(\alpha) = [\bar{\delta}, \alpha] := \bar{\delta} \circ \alpha + (-1)^{p+q+1} \alpha \circ \bar{\delta}\ .$$

Every *holomorphic* bundle $\mathcal{E}$ over X of differentiable type E induces a canonical semiconnection $\bar{\delta} := \bar{\partial}_{\mathcal{E}}$ on E such that $\bar{\delta}^2 : A^0(E) \longrightarrow A^{0,2}(E)$ vanishes identically. Conversely, by [1], Theorem (5.1), every semiconnection $\bar{\delta}$ with $\bar{\delta}^2 = 0$ defines a unique holomorphic bundle $\mathcal{E}$, differentiably equivalent to E such that $\bar{\partial}_{\mathcal{E}} = \bar{\delta}$.

There exists a natural right action of the gauge group $\mathrm{GL}(E) \subset A^0(\operatorname{End} E)$ of differentiable automorphisms of E on the space $\bar{\mathcal{D}}(E) \times A^0(E)$ by

$$(\bar{\delta}, \phi) \cdot g := (g^{-1} \circ \bar{\delta} \circ g, g^{-1}\phi)\ .$$

Denote by $\bar{\mathcal{S}}(E)$ the set of points with trivial isotropy group. After suitable L^2_k-Sobolev completions, the space $\bar{\mathcal{B}}^{\mathrm{s.p.}}(E) := \bar{\mathcal{S}}(E)/\,\mathrm{GL}(E)$ becomes a complex analytic Hilbert manifold, and $\bar{\mathcal{S}}(E) \longrightarrow \bar{\mathcal{B}}^{\mathrm{s.p.}}(E)$ a complex analytic $\mathrm{GL}(E)$-Hilbert principal bundle. The map

$$\Upsilon : \bar{\mathcal{D}}(E) \times A^0(E) \longrightarrow A^{0,2}(\operatorname{End} E) \times A^{0,1}(E)$$

given by

$$\Upsilon(\bar{\delta}, \phi) = (\bar{\delta}^2, \bar{\delta}\phi)$$

is $\mathrm{GL}(E)$-equivariant, hence it induces a section $\hat{\Upsilon}$ in the associated Hilbert vector bundle

$$\bar{\mathcal{S}}(E) \times_{\mathrm{GL}(E)} \left[A^{0,2}(\operatorname{End} E) \oplus A^{0,1}(E)\right]$$

over $\bar{\mathcal{B}}^{\mathrm{s.p.}}(E)$. This section becomes analytic for apropriate Sobolev completions.

Definition 2.2. The *gauge-theoretical* moduli space $\mathcal{M}^{\mathrm{s.p.}}(E)$ of simple holomorphic pairs of type E is the complex analytic space given by the vanishing locus of the section $\hat{\Upsilon}$.

Set-theoretically, $\mathcal{M}^{s.p.}(E)$ can be identified with the set of isomorphism classes of pairs $(\mathcal{E},\phi)$, where $\mathcal{E}$ is a holomorphic bundle of type E, and ϕ is a holomorphic section in $\mathcal{E}$, such that the associated evaluation map

$$ev(\phi) : H^0(X, \mathcal{E}nd(\mathcal{E})) \longrightarrow H^0(X,\mathcal{E})$$

is injective. This is equivalent to the fact that the only automorphism of $(\mathcal{E},\phi)$ is the identity. If $\mathcal{E}$ is a *simple bundle* (this happens always if $r = 1$) and $\phi \in H^0(X,\mathcal{E})$, then $(\mathcal{E},\phi)$ is a *simple pair* iff ϕ is nontrivial.

Definition 2.3. Fix $(\bar{\delta}_0, \phi_0) \in \Upsilon^{-1}(0)$. A *gauge-theoretical family of deformations* of $(\bar{\partial}_0, \phi_0)$ parametrized by a germ $(T, 0)$ is a complex analytic map

$$\omega = (\omega_1, \omega_2) : (T,0) \longrightarrow (A^{0,1}(\operatorname{End} E), 0) \times (A^0(E), \phi_0)$$

such that the image of the map $(\bar{\delta}_0 + \omega_1, \omega_2)$ is contained in $\Upsilon^{-1}(0)$, and

$$(\bar{\delta}_0 + \omega_1, \omega_2) : (T,0) \longrightarrow \Upsilon^{-1}(0)$$

is also holomorphic.

Two families ω and ω' over $(T,0)$ are called *equivalent* if there exists a complex analytic map

$$g : (T,0) \longrightarrow (\mathrm{GL}(E), \mathrm{id}_E)$$

such that $\omega' = \omega \cdot g$.

Note that, given such a deformation $\omega = (\omega_1, \omega_2)$, the family ω induces uniquely a section in the sheaf $\big(\mathcal{A}^{0,1}(\operatorname{End} E) \times \mathcal{A}^{0,0}(E)\big)\hat{\otimes}\mathcal{O}_T$, and conversely. In particular, if $(T,0)$ is an *artinian* germ, then ω_1 induces a morphism of sheaves

$$\mathcal{A}^{0,i}(E)_T := \mathcal{A}^{0,i}(E) \otimes_{\mathbb{C}} \mathcal{O}_T \longrightarrow \mathcal{A}^{0,i+1}(E)_T$$

We denote by $h_{\text{gauge}} : (germs) \longrightarrow (sets)$ the functor which sends a germ $(T,0)$ into the set of equivalence classes of gauge-theoretical families of deformations over $(T,0)$.

Theorem 2.4. *The functor h_{gauge} has a semi-universal deformation.*

Proof. Fix $\bar{\delta}_0 \in \bar{\mathcal{D}}(E)$ and consider the orbit map $\beta : \mathrm{GL}(E) \longrightarrow \bar{\mathcal{D}}(E) \times A^0(E)$ given by

$$\beta(g) := (\bar{\delta}_0, 0) \cdot g = (\bar{\delta}_0 + g^{-1}\bar{D}_0(g), 0)\ .$$

By [4], Theorem (12.13) and [10], Theorem (1.1), the existence of a semi-universal deformation follows if:

i.) The derivative of Υ at $(\bar{\delta}_0, 0)$ and the derivative of β at id_E are direct linear maps (for apropriate Sobolev completions);

ii.) The quotient $\ker(d\Upsilon_{(\bar{\delta}_0,0)})\big/\operatorname{im}(d\beta_{\mathrm{id}_E})$ is finite dimensional.

One has

$$\ker(d\Upsilon_{(\bar{\delta}_0,0)}) = \Big\{\ (\alpha,\phi) \in A^{0,1}(\operatorname{End} E) \times A^0(E)\ \Big|\ \bar{D}_0(\alpha) = 0, \bar{\delta}_0\phi = 0\ \Big\}$$

and $d\beta_{\mathrm{id}_E} : A^0(\operatorname{End} E) \longrightarrow A^{0,1}(\operatorname{End} E) \times A^0(E)$ is given by $u \longmapsto (\bar{D}_0(u), 0)$. Therefore *i.*) and *ii.*) follow from standard Hodge theory. □

Remark. The existence of a moduli space of simple holomorphic pairs for arbitrary rank can be also deduced from [11], Theorem (2.1), (2.2) and the proof of Theorem (6.4) of loc.cit.

Indeed, one has local semi-universal deformations: Fix $(\mathcal{E}_0, \phi_0)$ and let $\mathcal{E} \longrightarrow X \times (R, 0)$ be a semi-universal family of vector bundles with $\mathcal{E}|_{X\times\{0\}} \cong \mathcal{E}_0$. Similarly as in rank one, the functor $\mathcal{H} : (an/R) \longrightarrow (sets)$ given by

$$\left(S \xrightarrow{f} R\right) \longmapsto \operatorname{Hom}\left(\mathcal{O}_{X\times S}, (\mathrm{id}_X \times f)^*\mathcal{E}\right)$$

is representable [8] by a linear fibre space $p : \tilde{R} \longrightarrow R$. Moreover, there exists a tautological section $u : \mathcal{O}_{X\times\tilde{R}} \longrightarrow (\mathrm{id}_X \times p)^*\mathcal{E}$ arising from the representability of $\mathcal{H}$. The pair $((\mathrm{id}_X \times p)^*\mathcal{E}, u)$ is a versal deformation of $(\mathcal{E}_0, \phi_0)$. By [3] there exists then also a semi-universal deformation.

Moreover, it can be shown (as in loc.cit. for the case of simple sheaves), that the isomorphy locus of two families of simple holomorphic pairs over S is a *locally closed* analytic subset of S and as a consequence, this moduli space exists by [11], (2.1).

The aim of the remaining part is to prove the following

Theorem 2.5. *The* gauge-theoretical *moduli space of simple holomorphic pairs of type E is analytically isomorphic to the* complex-theoretical *moduli space of simple holomorphic pairs of type E.*

Proof. The arguments we use are inspired from [12], where a similar problem is treated (bundles without section). It suffices to show that the associated deformation functors h_{gauge} resp. h_{an} are isomorphic over *artinian* bases.

Note first, that there exists a well defined morphism of functors from h_{an} to h_{gauge}. Moreover, this morphism is injective since

$$(\mathcal{E}_1, \phi_1) \sim (\mathcal{E}_2, \phi_2) \Longleftrightarrow (\bar{\partial}_{\mathcal{E}_1}, \phi_1) \sim (\bar{\partial}_{\mathcal{E}_2}, \phi_2)\ .$$

In order to prove the surjectivity, we need to show that every gauge-theoretical family of simple holomorphic pairs determines a complex-theoretical family of simple holomorphic pairs. We will prove this by induction on the length of the artinian base.

For $n = 0$, this follows from the fact that every integrable semi-connection $\bar{\delta}_0$ determines a holomorphic bundle $\mathcal{E}_0$ of type E such that one has an exact sequence of sheaves

$$0 \to \mathcal{E}_0 \to \mathcal{A}^{0,0}(E) \overset{\bar{\partial}_{\mathcal{E}_0}}{\to} \mathcal{A}^{0,1}(E) \overset{\bar{\partial}_{\mathcal{E}_0}}{\to} \mathcal{A}^{0,2}(E) \to \dots$$

For the induction step, let $(T, 0)$ be a small extension of an infinitesimal neighbourhood $(T', 0)$ such that $\ker(\mathcal{O}_T \longrightarrow \mathcal{O}_{T'}) = \mathbb{C}$, and let $\omega = (\omega_1, \omega_2)$ be a gauge-theoretical family of simple holomorphic pairs parametrized by $(T, 0)$. By the induction assumption, we can find a holomorphic vector bundle $\mathcal{E}'$ over $X \times (T', 0)$

which is induced by $\omega_1|_{T'}$. Then we have the following exact sequence of sheaves

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{E}_0 & \to & \mathcal{A}^{0,0}(E) & \to & \mathcal{A}^{0,1}(E) & \to & \mathcal{A}^{0,2}(E) & \to & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{E} & \to & \mathcal{A}^{0,0}(E)_T & \to & \mathcal{A}^{0,1}(E)_T & \to & \mathcal{A}^{0,2}(E)_T & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{E}' & \to & \mathcal{A}^{0,0}(E)_{T'} & \to & \mathcal{A}^{0,1}(E)_{T'} & \to & \mathcal{A}^{0,2}(E)_{T'} & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

In particular, $\mathcal{E}$ is locally free, hence it defines a holomorphic vector bundle over $X \times (T, 0)$ whose restriction to $X \times (T', 0)$ gives $\mathcal{E}'$. This vector bundle $\mathcal{E}$ together with the family of sections ω_2 gives rise to a complex theoretical family of simple pairs which induces the gauge-theoretical family ω.

This completes the proof of the theorem. □

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